Relativistic mass increase at slow speeds

Gerald Gabrielse
Department of Physics, Harvard University, Cambridge, Massachusetts 02138

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I. MASSES OF BOTTLED ANTI-PROTONS

It is now possible to cool antiprotons to a few kelvins and hold them in an ion trap for several months.[1] Such ion traps permit remarkably precise mass spectrometry. For example, when 400- and 200-eV antiprotons are held simultaneously in the same trap, it is easy to observe and measure their relativistic mass difference even though their speeds differ by less than 0.03% of c.

The following problem is suitable as an exercise in the dynamics of special relativity. A student needs to be familiar with the idea of cyclotron resonance and how mass changes as a function of velocity.

II. BACKGROUND

Because charged particles of velocity v spiral around magnetic field lines, a magnetic field B will confine the particles in two dimensions. Confinement in the third direction can be achieved by an electrostatic quadrupole field. The combination makes up what is known as a Penning trap.[2,3]

The electric and magnetic fields in the trap keep the ions from touching the container walls, so such a trap can be used to hold anti-matter. It is also necessary that there be very few atoms of ordinary matter in the trap, otherwise the antiprotons will collide with them and annihilate. In 1990, in a vacuum of less than 5×10^{-17} Torr, about 10^3 antiprotons were held in such a trap for over 2 months.[1,4,5]

In addition to their thermal motion, ions in such traps have regular periodic motions. The principal motion around the magnetic field lines at the cyclotron frequency. The cyclotron frequency ν_c derived from the Lorentz force is

$$\nu_c=\frac{qB}{2\pi\gamma m},$$

where q is the electric charge of a particle with an invariant mass m moving in a uniform magnetic field B, and γ is the usual relativistic factor $1/(\sqrt{1-v^2/c^2})$.

For many problems involving the cyclotron frequency the particle velocity is so low that for practical purposes $\gamma=1$, and it is omitted from the formula. However, the cyclotron frequency of ions in an ion trap can be measured with such great precision that the effect of the relativistic mass increase can be detected when γ deviates from unity by only a few parts in 10^{-9}.

III. THE PROBLEMS

A. Cooled antiprotons

In a Penning trap, antiprotons act like a gas with two degrees of freedom that can quickly reach thermal equilibrium with their environment. In this experiment, the environment is at the temperature of liquid helium, ~4 K. What is the average thermal kinetic energy of these antiprotons?

B. Cyclotron frequency

Discussions of the cyclotron often emphasize that the cyclotron frequency ν_c is independent of the energy of the accelerated particle. Show that when you take special relativity into account this is not true and that for $v/c\ll1$ to a good approximation

$$\nu_c=\frac{qB}{2\pi m}\left[1-\frac{K}{mc^2}\right],$$

where K is the kinetic energy of the particle.

What kinetic energy will change γ by an amount equal to 10^{-9}?

Derive an expression for the fractional change of ν_c corresponding to a small change dK in K.

C. Some real data

Two antiprotons cooled to about 4 K in a trap exhibit two different cyclotron frequencies as shown in Fig. 1. From the data given in the figure determine the magnetic field of the trap, the difference in the energies of the two antiprotons, and their individual kinetic energies. The zero point on the abscissa corresponds to the resonance of an antiproton if it had had no relativistic increase in energy.

D. Distinguishing thermal from mechanical energy

How is it that two particles that differ in kinetic energy by 192 eV can be said to have a temperature of only 4 K?

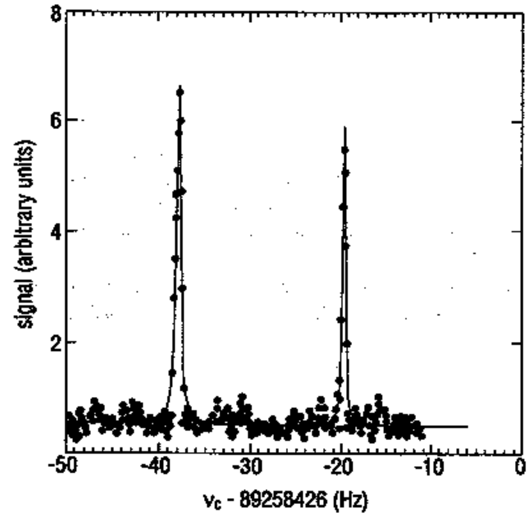

Fig. 1. Frequencies of cyclotron resonances of two antiprotons in a Penning trap. The zero point on the abscissa corresponds to the resonance of an antiproton if it had had no relativistic mass increase.

IV. SOLUTIONS

A. Cooled antiprotons

For particles with only two degrees of freedom the average thermal kinetic energy $\bar{K}_t$ is just k_BT where $k_B=1.38\times10^{-23}$ J/K is the Boltzmann constant and T is the temperature of the particles in kelvins. You can work this out by substituting numbers into the formula and converting them to eV:

$$\bar{K}_t=\frac{1.38\times10^{-23}}{1.602\times10^{-19}}\,4=3.44\times10^{-4}\text{ eV}.$$

Of course, you can find the answer more easily if you remember that at room temperature k_BT is 1/40 eV. Then at 4 K, $k_BT=\frac{1}{40}\frac{4}{293}=3.4\times10^{-4}$ eV.

B. Cyclotron frequency

For low energies it is convenient to show the effect of relativistic mass increase by expanding $1/\gamma$ to first order in v^2/c^2. Then,

$$\nu_c=\frac{qB}{2\pi m}\left[1-\frac{v^2}{2c^2}\right].$$

Multiplying numerator and denominator of the second term in the expansion by m gives $mv^2/2mc^2$ and you can identify $mv^2/2$ as the kinetic energy K which gives

$$\nu_c=\frac{qB}{2\pi m}\left[1-\frac{K}{mc^2}\right]. \tag{1}$$

The value of $\gamma-1$ is K/mc^2. For an antiproton, this will be 10^{-9} when

$$K=mc^2 10^{-9}=0.938\times10^9\times10^{-9}=0.94\approx1\text{ eV}.$$

To derive an expression for the fractional change $d\nu_c/\nu_c$ of the cyclotron frequency produced by a change in kinetic energy dK, differentiate Eq. (1) and divide the result by Eq. (1). This gives

$$\frac{d\nu_c}{\nu_c}=-\frac{dK}{mc^2-K}\approx-\frac{dK}{mc^2}. \tag{2}$$

C. Some real data

The graph in Fig. 1 shows measured cyclotron resonances of two antiprotons in a Penning trap. The peaks are within a few hertz of the frequency 89.258 426 MHz. From Eq. (1), it follows that

$$B=\frac{89.26\times10^6\times2\pi\times938\times10^6}{9\times10^{16}}=5.85T.$$

From the graph we see that the two peaks differ in frequency by 18.3 Hz. From Eq. (2) it follows that

$$dK=\frac{18.3}{89.26\times10^6}\,938.3\times10^6=192\text{ eV}.$$

The above calculation has the virtue that it can be done without knowledge of the frequency corresponding to the exact rest mass. If you know that that frequency is 89.258 426 MHz, you can use the information from Fig. 1 to calculate the kinetic energy of each antiproton. From Eq. (1) it is easy to see that

$$K=\frac{(\nu_c-89\ 258\ 426)}{89.26\times10^6}\,938.3\times10^6$$

$$=(\nu_c-89\ 258\ 426)10.51\text{ eV}.$$

For the righthand peak in Fig. 1, $\nu_c-89\,258\,426=19.6$ Hz. This yields a kinetic energy of $K_r=206$ eV. The left-hand peak is at 37.9 Hz which corresponds to $K_l=398$ eV.

D. Distinguishing thermal from mechanical energy

The temperature T is a measure of the energy of an ion's random fluctuations about its average speed as it travels along its trajectory in the trap. Two ions with well defined, different speeds in their trajectories will have quite different values of kinetic energy K, but if they have the same very small random fluctuations about their average speeds, they will have the same very low temperature.

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