Chapter 6

Determining the Fine Structure Constant

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The most accurate determination of the fine structure constant α is $\alpha^{-1} = 137.035\,999\,084\,(51)\,[0.37\,\text{ppb}]$. This value is deduced from the measured electron $g/2$ (the electron magnetic moment in Bohr magnetons) using the relationship of α and $g/2$ that comes primarily from Dirac and QED theory. Less accurate by factors of 12 and 21 are determinations of α from combined measurements of the Rydberg constant, two mass ratios, an optical frequency, and a recoil shift for Rb and Cs atoms. Helium fine structure intervals have been measured well enough to determine α with nearly the same precision – if two-electron QED calculations can be sorted out. Less accurate measurements are also compared.

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6.1. Introduction

The fundamental and dimensionless fine structure constant α is defined (in SI units) by

$$\alpha = \frac{1}{4\pi\epsilon_0} \frac{e^2}{\hbar c}.$$ (6.1)

The well known value $\alpha^{-1} \approx 137$ is not predicted within the Standard Model of particle physics.

The most accurate determination of α comes from a new Harvard measurement [7, 8] of the dimensionless electron magnetic moment, $g/2$, that is 15 times more accurate than the measurement that stood for twenty years [9]. The fine structure constant is obtained from $g/2$ using the theory of a Dirac point particle with QED corrections [10–15]. The most accurate α, and the two most accurate independent values, are given by

$$\alpha^{-1}(\text{H08}) = 137.035\ 999\ 084\ (51) \qquad [0.37\ \text{ppb}] \qquad (6.2)$$
$$\alpha^{-1}(\text{Rb08}) = 137.035\ 999\ 45\ (62) \qquad [4.5\ \text{ppb}] \qquad (6.3)$$
$$\alpha^{-1}(\text{Cs06}) = 137.036\ 000\ 0\ (11) \qquad [8.0\ \text{ppb}]. \qquad (6.4)$$
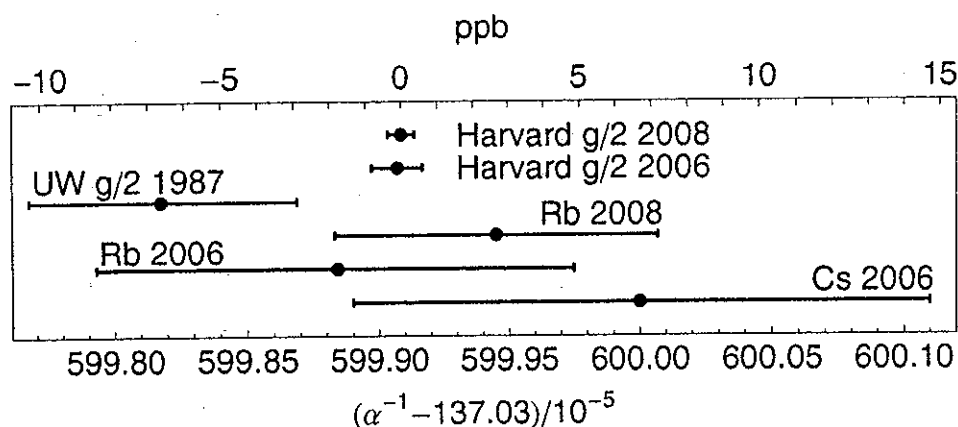
Fig. 6.1 compares the most accurate values.

Fig. 6.1. The most precise determinations of α.

The uncertainties in the two independent determinations of α are within a factor of 12 and 21 of the α from $g/2$. They rely upon separate measurements of the Rydberg constant [16, 17], mass ratios [18, 19], optical frequencies [20, 21], and atom recoil [21, 22]. Theory also plays an important role for this method, to determine the Rydberg constant (reviewed in Ref. [23]) and one of the mass ratios [24].

In what follows, the importance of the fine structure constant is discussed first. Determining α from the measured electron $g/2$ comes next, starting with an operational summary of how this is done, and finishing with an overview of the status and reliability of the theory. Determining α from the combined measurements mentioned above is the next topic. The possibility to determine α with nearly the same precision from atomic fine structure is then considered. Helium fine structure intervals have been measured with enough accuracy to do so, [1–4, 25] if inconsistencies in the needed two-electron QED theory [5, 6] can be cleared up. Other methods that are important for historical reasons are mentioned, and followed by a conclusion.

6.2. Importance of the Fine Structure Constant

The fine structure constant appears in many contexts and is important for many reasons.

(1) The fine structure constant is the low energy electromagnetic coupling constant, the measure of the strength of the electromagnetic interaction in the low energy limit.

(2) The fine structure constant is the basic dimensionless constant of atomic physics, distinguishing the energy scales that are important for atoms. In terms of the electron rest energy, $m_e c^2$:

   (a) The binding energy of an atom is approximately $\alpha^2 m_e c^2$.
   (b) The fine structure energy splitting in atoms goes as $\alpha^4 m_e c^2$.
   (c) The hyperfine structure energy splitting goes as
       $(m_e/M) \alpha^4 m_e c^2$, like the fine structure splitting except reduced by an additional ratio of an electron mass to the nucleon mass (M).
   (d) The lamb shift in an atom goes as $\alpha^5 m_e c^2$.

(3) The fine structure constant is also important for condensed matter physics, the condensed matter and atomic energy scales being similar. Important examples include the quantum hall resistance and the oscillation frequency of a Josephson junction.

(4) The fine structure constant is important and central to our interlinked system of fundamental constants [23]. Its role will be enhanced if a contemplated redefinition of the SI system of units

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(to remove the dependence upon an artifact mass standard) is adopted [27].

(5) Measurements of the muon magnetic moment [28], made to test for possible breakdowns of the Standard Model of particle physics, require a value for α. Small departures from the Standard Model would only be visible once the large α-dependent QED contribution to the muon g value is subtracted out.

(6) Comparing α values from methods that depend differently upon QED theory is a test of the QED theory.

6.3. Most Accurate α Comes from Electron $g/2$

6.3.1. *New Harvard measurement and QED theory*

The most accurate determination of the fine structure constant utilizes a new measurement of the electron magnetic moment, measured in Bohr magnetons [7],

$$g/2 = 1.001\,159\,652\,180\,73\,(28)\,[0.28\,\text{ppt}]. \tag{6.5}$$

This 2008 measurement of $g/2$ (Chapter 5) is 15 times more precise than the 1987 measurement [9] that had stood for about twenty years. The high precision and accuracy came from new methods that made it possible to resolve the quantum cyclotron levels [29], as well as the spin levels, of one electron suspended for months at a time in a cylindrical Penning trap [30].

The electron $g/2$ is essentially the ratio of the spin and cyclotron frequencies. This ratio is deduced from measurable oscillation frequencies in the trap using an invariance theorem [31]. These frequencies are measured using quantum jump spectroscopy of one-quantum transitions between the lowest energy levels [8]. The cylindrical Penning trap electrodes form a microwave cavity that shapes the radiation field in which the electron is located, narrowing resonance linewidths by inhibiting spontaneous emission [29, 32], and providing boundary conditions which make it possible to identify the symmetries of cavity radiation modes [7, 33]. A QND (quantum nondemolition) coupling, of the cyclotron and spin energies to the frequency of an orthogonal and nearly harmonic electron axial oscillation, reveals the quantum state [29]. This harmonic oscillation of the electron is self-excited [34], by a feedback signal [35] derived from its own motion, to produce the large signal-to-noise ratio needed to quickly read out the quantum state without ambiguity.

Within the Standard Model of particle physics the measured electron $g/2$ is related to the fine structure constant by

$$g/2 = 1 + C_2\left(\frac{\alpha}{\pi}\right) + C_4\left(\frac{\alpha}{\pi}\right)^2 + C_6\left(\frac{\alpha}{\pi}\right)^3 + C_8\left(\frac{\alpha}{\pi}\right)^4 + C_{10}\left(\frac{\alpha}{\pi}\right)^5$$
$$+ \ldots + a_{\text{hadronic}} + a_{\text{weak}}. \tag{6.6}$$

Dirac theory of the electron provides the leading term on the right. Fig. 6.2 compares the size of the measured $g/2$ (gray) with its measurement uncertainty (black) to size of this leading Dirac term and other theoretical contributions (gray). The uncertainties (black) of the theoretical contributions arise from the uncertainty for the coefficients.
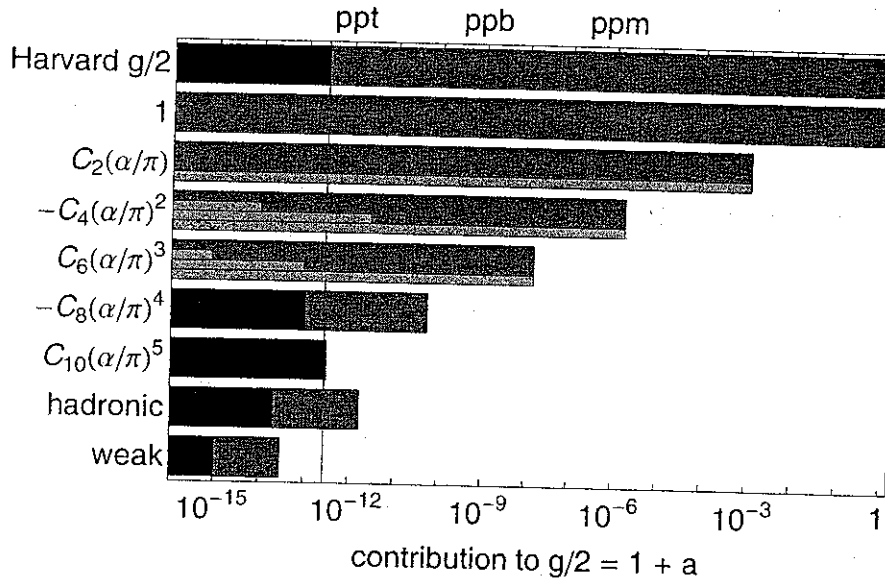

Fig. 6.2. Contributions to $g/2$ for the experiment (top bar), terms in the QED series (below), and from small distance physics (below). Uncertainties are black. The inset light gray bars represent the magnitude of the larger mass-independent terms (A_1) and the smaller A_2 terms that depend upon either m_e/m_μ or m_e/m_τ. The even smaller A_3 terms, functions of both mass ratios, are not visible on this scale.

Quantum electrodynamics (QED) provides the expansion in the small ratio $\alpha/\pi \approx 2 \times 10^{-3}$, and the values of the coefficients C_k. The first three of these, C_2 [10], C_4 [11–13], C_6 [14] are exactly known functions which have no theoretical uncertainty. The small uncertainties in C_4 and C_6, completely negligible at the current level of experimental precision (Fig. 6.2), arise

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because C_4 and C_6 depend slightly upon lepton mass ratios.

$$C_2 = \quad 0.500\ 000\ 000\ 000\ 00 \text{ (exact)} \tag{6.7}$$

$$C_4 = -\ 0.328\ 478\ 444\ 002\ 90\ (60) \tag{6.8}$$

$$C_6 = \quad 1.181\ 234\ 016\ 827\ (19) \tag{6.9}$$

$$C_8 = -\ 1.914\ 4\ (35) \tag{6.10}$$

$$C_{10} = \quad 0.0\ (4.6). \tag{6.11}$$

There is no analytic solution for C_8 yet but this coefficient has been calculated numerically [15]. Unfortunately, C_{10} has not yet been calculated; the quoted bound is a simple extrapolation from the lower-order C_k [36].

Very small additional contributions due to short distance physics have also been evaluated [37, 38],

$$a_{\text{hadronic}} = \quad 0.000\ 000\ 000\ 001\ 682\ (20) \tag{6.12}$$

$$a_{\text{weak}} = \quad 0.000\ 000\ 000\ 000\ 030\ (01). \tag{6.13}$$

The hadronic contribution is important at the current level of experimental precision, but the reported uncertainty for this contribution is much smaller than is currently needed to determine α from $g/2$. See Chapters 8 and 9 for further details.

The most precise value of the fine structure constant comes from using the very accurately measured electron $g/2$ (Eq. (6.5)) in the Standard Model relationship between $g/2$ and α (Eq. (6.6)). The result is

$$
\begin{aligned}
\alpha^{-1}(\text{H08}) &= 137.035\ 999\ 084\ (33)\ (39) & &[0.24\text{ ppb}]\ [0.28\text{ ppb}], \\
&= 137.035\ 999\ 084\ (33)\ (12)\ (37) & &[0.24\text{ ppb}]\ [0.09\text{ ppb}]\ [0.27\text{ ppb}], \\
&= 137.035\ 999\ 084\ (51) & &[0.37\text{ ppb}]. \tag{6.14}
\end{aligned}
$$

The first line shows experimental (first) and theoretical (second) uncertainties that are nearly the same. The second line separates the theoretical uncertainty into two parts, the numerical uncertainty in C_8 (second) and the estimated uncertainty for C_{10} (third). The third line gives the total 0.37 ppb uncertainty. A graphical comparison of the experimental and theoretical uncertainties in determining α from $g/2$ is in Fig. 6.3.

The crudely estimated theoretical uncertainty in the uncalculated C_{10} currently adds more to the uncertainty in α more than does the measurement uncertainty for $g/2$. As a result, the factor of 15 reduction in the measurement uncertainty for $g/2$ results in only a factor of 10 reduction in the uncertainty in α.
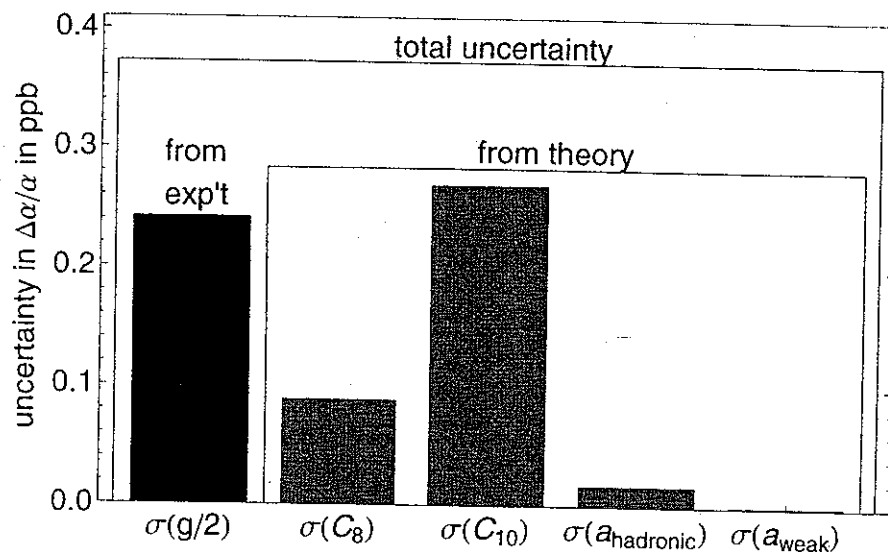
Fig. 6.3. Experimental uncertainty (black) and theoretical uncertainties (gray) that determine the uncertainty in the α that is determined from the measured electron $g/2$.

Figure 6.1 compares our $\alpha^{-1}(H08)$ to other accurate determinations of α. The fine structure constant is currently determined about 12 and 21 times more precisely from $g/2$ than from the best Cs and Rb measurements (to be discussed). No other α determination has error bars small enough to fit in this figure. Comparing our α with the most accurate independent determinations is a test of the Standard Model prediction in Eq. (6.6), along with the theoretical assumptions used for the other determinations. More accurate independent α values would improve upon what is already the most stringent test of QED theory.

6.3.2. *Status and reliability of the QED theory*

The electron $g/2$ differs from 1 by about one part in 10^3 as a result of the QED corrections to the Dirac theory. How uncertain and how reliable is the QED theory that is needed to accurately determine α from $g/2$? Given the complexity of the theory, and mistakes that have been discovered in the past, how likely is it that additional mistakes will either appreciably change α in the future, or go undetected?

In this section we summarize the status of calculations of the C_k coefficients, the current values of which are already listed in Eqs. 6.7–6.11. The history and method of the calculations are discussed in Chapters 3 and 4. We illustrate how impressive analytic calculations have made it easy to now

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evaluate the lowest order coefficients (C_2, C_4 and C_6) to an arbitrary precision with no theoretical uncertainty, provided that no mistakes have been made. Numerical calculations and verifications of C_8, and the prospects for numerical calculations of C_{10}, are also summarized.

There is no theoretical uncertainty in the Dirac unit contribution to $g/2$ in Eq. (6.6). There is also no theoretical uncertainty in the leading QED correction, $C_2(\alpha/\pi)$, insofar as long ago a single Feynman diagram was evaluated analytically to determine C_2 exactly [10].

The C_4 coefficient is the sum of a mass-independent term and two much smaller terms that are functions of lepton mass ratios,

$$C_4 = A_1^{(4)} + A_2^{(4)}(\frac{m_e}{m_\mu}) + A_2^{(4)}(\frac{m_e}{m_\tau}). \tag{6.15}$$

The mass-independent term is larger by many orders of magnitude. This pure number, involving 7 Feynman diagrams, is given by [11–13, 39]

$$A_1^{(4)} = \frac{197}{144} + \frac{\pi^2}{12} + \frac{3}{4}\zeta(3) - \frac{\pi^2}{2}\ln(2) \tag{6.16}$$

$$= -0.328\,478\,965\,579\,193\ldots \tag{6.17}$$

where $\zeta(s)$ is the Riemann zeta function (Zeta[s] in Mathematica).

There is no theoretical uncertainty in this contribution, which can easily be evaluated to any desired precision. Of course, this is only true if there are no mistakes in the analytic derivation. The original result [40] had an error in the evaluation of an integral. This was corrected some years later [12] (and then confirmed independently [11]) after the initial result did not agree with a numerical calculation. This was the first of several instances where independent evaluations allowed the elimination of mistakes, as we shall see.

The mass-dependent function $A_2^{(4)}(x)$ is an analytical evaluation of one Feynman diagram [41]. In a convenient form [42] it is given by

$$A_2^{(4)}(1/x) = -\frac{25}{36} - \frac{\ln(x)}{3} + x^2[4 + 3\ln(x)] + \frac{x}{2}(1 - 5x^2)$$
$$\times \left[\frac{\pi^2}{2} - \ln(x)\ln(\frac{1-x}{1+x}) - \text{Li}_2(x) + \text{Li}_2(-x)\right]$$
$$+ x^4\left[\frac{\pi^2}{3} - 2\ln(x)\ln(\frac{1}{x} - x) - \text{Li}_2(x^2)\right]. \tag{6.18}$$

The dilogarithm function is a special case of the polylogarithm (Poly-Log[n,x] in Mathematica); it has a series expansion $\text{Li}_n(x) = \sum_{k=1}^{\infty} x^k/k^n$

that converges for the cases we need. The exactly calculated mass-dependent function is evaluated as a function of two lepton mass ratios [23, 43],

$$m_\mu/m_e = 206.768\ 276\ (24) \tag{6.19}$$

$$m_\tau/m_e = 3\ 477.48\ (57). \tag{6.20}$$

There is no theoretical uncertainty in the mass-dependent terms

$$A_2^{(4)}(m_e/m_\mu) = 5.197\ 387\ 71\ (12) \times 10^{-7}, \tag{6.21}$$

$$A_2^{(4)}(m_e/m_\tau) = 1.837\ 63\ (60) \times 10^{-9}. \tag{6.22}$$

The uncertainties are from the uncertainties in the measured mass ratios. When multiplied by $(\alpha/\pi)^2$ these are very small contributions to $g/2$. The first of these two contributions is larger than the current experimental precision (Fig. 6.2) while the second is not. The uncertainties in both terms are so small as to not even be visible in Fig. 6.2.

The higher order coefficients, C_k with $k = 6, 8, 10, \ldots$, are each the sum of a constant and functions of mass ratios,

$$C_k = A_1^{(k)} + A_2^{(k)}\left(\frac{m_e}{m_\mu}\right) + A_2^{(k)}\left(\frac{m_e}{m_\tau}\right) + A_3^{(k)}\left(\frac{m_e}{m_\mu}, \frac{m_e}{m_\tau}\right). \tag{6.23}$$

The leading mass-independent term, $A_1^{(k)}$, is much larger than the small mass-dependent corrections. In fact, for $k \geq 8$, the mass-dependent corrections should not be needed to determine α from $g/2$ at the current or foreseeable measurement precision in $g/2$ owing to their very small values.

For sixth order the mass-independent term requires the evaluation of 72 Feynman diagrams. An analytic evaluation of this term, mostly by Remiddi and Laporta [14], is

$$
\begin{aligned}
A_1^{(6)} &= \frac{83}{72}\pi^2\zeta(3) - \frac{215}{24}\zeta(5) - \frac{239}{2160}\pi^4 + \frac{28259}{5184} \\
&\quad + \frac{139}{18}\zeta(3) - \frac{298}{9}\pi^2\ln(2) + \frac{17101}{810}\pi^2 \\
&\quad + \frac{100}{3}\left[\mathrm{Li}_4(\tfrac{1}{2}) + \frac{\ln^4(2)}{24} - \frac{\pi^2\ln^2(2)}{24}\right]
\end{aligned}
\tag{6.24}
$$

$$= 1.181\ 241\ 456\ 587\ \ldots. \tag{6.25}$$

This remarkable analytic expression, easily evaluated to any desired numerical precision with no theoretical error, is very significant for determining α from $g/2$ insofar as it completely removes what otherwise would be a significant numerical uncertainty.

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Is the remarkable analytic expression free of mistakes? The best confirmation is the good agreement between the extremely complicated analytic derivation and a simpler but computation-intensive numerical calculation, $A_1^{(6)} = 1.181\ 259\ (40)$ [44]. This result used the best computers available many years ago; it could (and should) now be greatly improved. An earlier numerical evaluation led to the discovery and correction of a mistake made in an earlier analytic derivation of a renormalization term [44]. This further illustrates the importance of checking analytic derivations numerically.

An exact analytic calculation of the 48 Feynman diagrams that determine the mass-dependent function $A_2^{(6)}$ has also been completed [45, 46]. However, the resulting expressions are apparently too lengthy to publish in a printed form. Instead, expansions for small mass ratios are made available

$$A_2^{(6)}(r) = \sum_{k=1}^{4} r^{2k} f_{2k}(r). \qquad (6.26)$$

The expansions make it easy to calculate the two most important mass dependent contributions to the precision at which the measurement uncertainty in the mass ratios is important for any foreseeable improvements in the mass ratio uncertainties. Functions f_2 and f_4 are from Ref. [46], f_6 is from Refs. [45] and [47], and f_8 is from Ref. [42].

$$f_2(r) = \frac{23\ln(r)}{135} + \frac{3\zeta(3)}{2} - \frac{2\pi^2}{45} - \frac{74957}{97200}, \qquad (6.27)$$

$$f_4(r) = -\frac{4337\ln^2(r)}{22680} + \frac{209891\ln(r)}{476280} + \frac{1811\zeta(3)}{2304} - \frac{1919\pi^2}{68040} \\ - \frac{451205689}{533433600}, \qquad (6.28)$$

$$f_6(r) = -\frac{2807\ln^2(r)}{21600} + \frac{665641\ln(r)}{2976750} + \frac{3077\zeta(3)}{5760} \\ - \frac{16967\pi^2}{907200} - \frac{246800849221}{480090240000}, \qquad (6.29)$$

$$f_8(r) = -\frac{55163\ln^2(r)}{594000} + \frac{24063509989\ln(r)}{172889640000} + \frac{9289\zeta(3)}{23040} \\ - \frac{340019\pi^2}{24948000} - \frac{896194260575549}{2396250410400000}. \qquad (6.30)$$

These expansions have been compared to the exact calculations to verify the claim that their accuracy is much higher than any experimental uncertainty that will likely be reached [42].

With the current values of the mass ratios,

$$A_2^{(6)}(m_e/m_\mu) = -7.373\ 941\ 58\ (28) \times 10^{-6}, \qquad (6.31)$$

$$A_2^{(6)}(m_e/m_\tau) = -6.581\ 9\ (19) \times 10^{-8}. \qquad (6.32)$$

The uncertainties arise from the measurement imprecision in the mass ratios, not from any theoretical uncertainty. The term that depends upon both mass ratios [42],

$$A_3^{(6)}(m_e/m_\mu, m_e/m_\tau) = 1.909\ 45\ (62) \times 10^{-13}, \qquad (6.33)$$

is too small to be important for the electron $g/2$ in the foreseeable future, or to even have its uncertainty visible in Fig. 6.2.

For the current and foreseeable experimental precisions, only the mass-independent term is required in eighth order. Kinoshita and his collaborators have reduced the 891 Feynman diagrams to a much smaller number of master integrals, which were then evaluated by Monte Carlo integrations over the course of ten years. The latest result is [15]

$$C_8 = A_1^{(8)} = -1.9144\ (35). \qquad (6.34)$$

The uncertainty is that of the numerical integration as evaluated by an integration routine [48], limited by the computer time available for the integrations. A calculation of this coefficient to 0.2% is a remarkable result that is critical for determining α from $g/2$.

Checking the eighth-order coefficient to make sure that it is correctly evaluated is a formidable challenge. There is no analytic result to compare (yet). Only the collaborating groups of Kinoshita and Nio have had the courage and tenacity needed to complete such a challenging calculation. The complexity of the calculation makes it very difficult to avoid mistakes. The strategy has been to check each part of the calculation by using more than one independent formulation [49].

Our 2006 measurement of $g/2$ came while the theoretical checking was underway. At this point we published a value of α along with a warning that the theoretical checking for eighth order was not yet complete [50]. In 2007, a calculation using an independent formulation reached a precision sufficient to reveal a mistake [15] in how infrared divergences were handled in two master integrals. When the mistake in C_8 was corrected, the α determined from $g/2$ shifted a bit [50].

One could take the moral of the 2007 adjustment to be that the sheer complexity of the high order QED calculation makes it impossible to be certain that they are done correctly. I take the opposite conclusion, choosing to be reassured that the theory is checked so carefully that even a very small mishandling of divergences can be identified and corrected. Now that the eighth order calculation is completely checked by an independent formulation, to a level of precision that the theorists deem is sufficient to detect

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mistakes, it seems much less likely that another substantial change in α will be necessary. The check will be even better when the new calculation reaches the numerical precision of the calculation being checked.

An evaluation of, or at least a reasonable bound on, the tenth-order coefficient, $C_{10} \approx A_1^{(10)}$, is needed as a result of the level of accuracy of our 2008 measurement of $g/2$. A calculation is not easy given that 12 672 Feynman diagrams contribute. The estimated bound suggested in the meantime [37],

$$C_{10} = 0.0\,(4.6), \tag{6.35}$$

takes the uncalculated coefficient to be zero with an uncertainty that is an extrapolation of the size of the lower order coefficients. This crude estimate is not so convincing. It is especially unsatisfying given that it now limits the accuracy with which α can be determined from the measured $g/2$, as illustrated in Fig. 6.3.

6.3.3. *How much better can α be determined?*

Fig. 6.3 shows the experimental and theoretical contributions to the uncertainty in the α determined from $g/2$. This uncertainty is currently divided nearly equally between measurement uncertainty in $g/2$ and theoretical uncertainty in the Standard Model relation between $g/2$ and α. The largest theoretical uncertainty is from the uncalculated C_{10}, followed by numerical uncertainty in C_8.

The first calculation of $C_{10} \approx A_1^{(10)}$ is now underway [15, 51, 52]. It has already produced an automated code that was checked by recomputing the eighth-order coefficient. (This is the independent calculation that in 2007 reached the precision needed to expose a mistake in the calculation of C_8 [15].) No limit or bound will apparently be available until the impressive calculation is completed at some level of precision because many contributions with similar magnitudes sum to make a smaller result. A completed calculation of C_{10} will likely reduce the theoretical uncertainty enough so that the uncertainty in α would approach the 0.26 ppb uncertainty that comes from the measurement uncertainty in $g/2$.

The uncertainty in C_8 can be reduced once the uncertainty in C_{10} has been reduced enough to warrant this. More computation time would reduce the numerical integration uncertainty in C_8. A better hope is that parts or all of this coefficient will eventually be calculated analytically. Efforts in this direction are underway [53].

It thus seems likely that the theoretical uncertainty that limits the accuracy to which α can be determined from $g/2$ can and will be reduced below

0.1 ppb. The corresponding good news is that it also seems likely that the uncertainty in α from the measurement of $g/2$ can also be reduced below 0.1 ppb. With enough experimental and theoretical effort it may well be possible to do even better.

6.4. Determining α from the Rydberg, Two Mass Ratios and $\hbar/M$ for an Atom

All the determinations of α whose uncertainty is not much larger than 20 times the uncertainty of the α from $g/2$ are compared in Fig. 6.1. The values not from $g/2$ in this figure do not come from a single measurement. Instead, each requires the determination of four quantities from a minimum of six precise measurements, each measurement contributing to the uncertainty in the α that is determined. Theory, including QED theory, is essential to determining two of the measured quantities.

The definitions for α and the Rydberg constant R_∞ taken together yield

$$\alpha^2 = \frac{4\pi}{c} R_\infty \frac{\hbar}{m_e}. \tag{6.36}$$

No accurate measurement of $\hbar/m_e$ for the electron is available. However, a precisely measured $\hbar/M_x$ for a Cs or Rb atom (of mass M_x) can be used along with two measurable mass ratios, $A_r(e)$ and $A_r(x)$,

$$\alpha^2 = \frac{4\pi}{c} R_\infty \frac{A_r(x)}{A_r(e)} \frac{\hbar}{M_x}. \tag{6.37}$$

The speed of light, c, is defined in the SI system of units.

The first of the needed mass ratios, $A_r(e) = 12 m_e/M(^{12}C)$, is the electron mass in atomic mass units (amu). The second is the mass of Cs or Rb in amu, $A_r(x) = 12 M(x)/M(^{12}C)$. Determining the Rydberg constant accurately requires the precise measurements of two hydrogen transition frequencies (and less accurate measurements of other quantities). Determining $\hbar/M_x$ for Cs and Rb requires the measurement of an optical frequency ω and an atom recoil velocity v_r, or equivalent recoil frequency shift, ω_r.

The fractional uncertainties that contribute to the uncertainty in α are listed in Table 6.1 for Cs, and in Table 6.2 for Rb, in order of increasing precision. Owing to the square in Eq. (6.37) the fractional uncertainty in α is half the fractional uncertainty of the contributing measurements.

The Rydberg constant describes the structure of a non-relativistic hydrogen atom in the limit of an infinite proton mass. Real hydrogen atoms, of course, have fine structure, Lamb shifts, and hyperfine structure. The proton has a finite mass. The Dirac energy eigenvalues must be corrected for

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Table 6.1.   Measurements determining $\alpha(Cs)$.

Measurement quantity	ppb	$\Delta\alpha/\alpha$ ppb	References
ω_r	15.	7.7	[[22]]
$A_r(e)$	0.4	0.2	[[23, 54]]
$A_r(Cs)$	0.2	0.1	[[18]]
ω	0.007	0.007	[[20]]
R_∞	0.007	0.004	[[16, 17, 23]]
Best $\alpha(Cs)$		8.0	[[22]]

Table 6.2.   Measurements determining $\alpha(Rb)$.

Measurement quantity	ppb	$\Delta\alpha/\alpha$ ppb	References
ω_r	9.1	4.6	[[21]]
$A_r(e)$	0.4	0.2	[[23, 54]]
$A_r(Rb)$	0.2	0.1	[[18]]
ω	0.4	0.4	[[21]]
R_∞	0.007	0.004	[[16, 17, 23]]
Best $\alpha(Rb)$		4.6	[[21]]

relativistic recoil, QED self-energy effects, and QED vacuum polarization. Corrections for nuclear polarization, nuclear size and nuclear self-energy are important at the precision with which transition energies can be measured.

The theory needed to determine the Rydberg constant from measurements is described in a seven-page section of Ref. [23] entitled "Theory relevant to the Rydberg constant." The accepted value of the Rydberg comes from a best fit of the measurements of a number of accurately measured hydrogen transitions [16, 17], the proton-to-electron mass ratio [19], the size of the proton, etc. to the intricate hydrogen theory for each of the hydrogen transitions, using more precisely measured values for every quantity that is not determined best by fitting. A full discussion of this process and a complete bibliography for all the measurements and calculations that make important contributions is beyond the scope of this work. Tables 6.1 and 6.2 thus show the currently accepted uncertainty for the Rydberg constant [23] rather than the uncertainties from all the contributing measurements.

The measured electron mass in amu, $A_r(e)$, relies equally upon precise measurements [19, 54] and upon bound state QED theory [24], using

$$\frac{m_e}{M} = \frac{g_{bound}}{2} \frac{1}{q/e} \frac{\omega_c}{\omega_s}, \tag{6.38}$$

where q/e is the integer charge of the ion in terms of one quantum of charge. Measurements are made using a $^{12}C^{5+}$ (or $^{16}O^{7+}$) ion trapped in a pair of open access Penning traps [55], a type of trap we developed for accurate measurements of q/m for an antiproton. Spin flips and cyclotron excitations are made in one trap and then transferred to the other for detection in a strong magnetic gradient. The spin frequency ω_s of the electron bound in an ion is measured. The cyclotron frequencies ω_c of the ion is deduced from the measurable oscillation frequencies of the trapped ion using the Brown–Gabrielse invariance theorem [31, 56]. This determination of the electron mass in amu could not take place without an extensive QED calculation of the g value of an electron bound into an ion [24]. A less accurate measurement of the electron mass in amu does not rely on QED theory [57]; it agrees with the more accurate method.

The needed mass ratios, $A_r(x)$, are from measurements [18] using isolated ions in a orthogonalized hyperbolic Penning trap [58], a trap design we developed to facilitate precise measurements. Ion cyclotron frequencies are deduced from oscillation frequencies of the ions in the trap using the same invariance theorem [31, 56]. Ion cyclotron energy is transferred to the axial motion using a sideband method that allows cyclotron information to be read out by a SQUID detector that is coupled to the axial motion of an ion in a trap. Ratios of ion frequencies give the ratios of masses in a simple and direct way that is insensitive to theory. Ratios to of M_x to the carbon mass, as needed to get amu, came from using ions like CO_2^+ and several hydrocarbon ions as reference ions.

The basic idea of the $\hbar/M_x$ measurements for Cs and Rb is that when an atom absorbs a quantum of light from a laser field, or is stimulated to emit a quantum of light into a laser field, then the atom recoils with a momentum $M_x v_r = \hbar k$, where for a laser field with angular frequency ω we have $k = \omega/c$. Thus $\hbar/M_x$ is determined by the measured optical frequency of the laser radiation, ω, and by the atom recoil velocity v_r. The latter can be accurately measured from the recoil shift ω_r in the resonance frequency caused by the recoil of the atom.

The laser frequency is measured a bit differently for Cs and Rb. For Cs the needed frequencies are measured with a precision of 0.007 ppb, much more accurately than will likely be needed for some time, using an optical

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comb to directly measure the frequency with respect to hydrogen maser and a Cs fountain clock [20]. For Rb, a diode laser is locked to a stable cavity, and its frequency is compared using an intermediate reference laser to that of a two-photon Rb standard [59].

The largest uncertainty in determining α using Eq. (6.37) is the uncertainty in measuring the atom recoil velocity v_r, or equivalently the recoil shift $\omega_r = \frac{1}{2}M_x v_r^2 / \hbar$. This measurement uncertainty is much larger than the measurement uncertainty in R_∞, $A_r(e)$, $A_r(x)$, and ω, and is thus the limit to the accuracy with which α can currently be determined by this method. The availability of extremely cold laser-cooled atoms has led to significant progress by two different research groups. First came a Cs measurement at Stanford [22] in 2002. More recently came 2006 and 2008 measurements of slightly higher precision with Rb atoms at the LKB in Paris [21, 59].

The Cs recoil measurement [22] and the most accurate of the Rb measurement [21] both measure the atom recoil using atom interferometry. The so-called Ramsey–Bordé spectrometer [60] configuration that is used in both cases was developed to apply Ramsey separated oscillator field methods at optical frequencies. Pairs of stimulated Raman $\pi/2$ pulses produced by counter-propagating laser beams [61] split the wave packet of a cold atom into two phase-coherent wave packets with different atom velocities. A series of N Raman π pulses then add recoil kicks to both parts of the atom wave packet. When a final pair of Raman $\pi/2$ pulses make it possible for the previously separated parts of the wave packet to interfere, the interference pattern reveals the energy difference, and hence the recoil frequency difference, for the wave packets in the two arms of the interferometer.

The measured phase difference that reveals v_r and ω_r goes basically as N, where N is the number of additional recoil kicks given to the wave packets in both arms of the spectrometer. The experiments differ in the way that they seek to make N as large as possible. The initial Cs measurement used a sequence of π pulses to achieve $N = 30$. The most accurate of the Rb measurements achieved $N = 1600$ using a series of Raman transitions with the frequency difference between the counter-propagating laser beams being swept linearly in time. This can equivalently be regarded as a type of Bloch oscillation within an accelerating optical lattice [62].

An improved apparatus is under construction in the hope of improving the 2008 measurement of the Rb recoil shift on the time scale of a year or two. Although no Cs recoil measurement has been reported since 2002, an improved apparatus has been built. A goal of soon measuring the Cs recoil

shift accurately enough to determine α to sub-ppb accuracy was mentioned in a recent report on improved beam splitters for a Cs atom interferometer [63].

6.5. Other Measurements to Determine α

6.5.1. *Determining α from He fine structure*

Surprisingly none of the accurate measurements determine α by measuring atomic fine structure intervals. Helium fine structure intervals have been measured precisely enough so that two-electron QED theory could determine α from the interval at about the same precision as do the combined Rydberg, mass ratios and atom recoil measurements. Helium is a better candidate for such measurements than is hydrogen because the fine structure splittings are larger, and the radiation lifetimes of the levels are longer so that narrow resonance lines can be measured. Unfortunately, theoretical inconsistencies need to be resolved.

The most accurate measurements of three 2^3P ^{4}He fine structure intervals [1–4, 25] are in good agreement as illustrated in Fig. 6.4. Our Harvard laser spectroscopy measurements [25] have the smallest uncertainties,

$$f_{12} = 2\,291\,175.59 \pm 0.51 \text{ kHz} \qquad [220 \text{ ppb}] \qquad (6.39)$$
$$f_{01} = 29\,616\,951.66 \pm 0.70 \text{ kHz} \qquad [\,24 \text{ ppb}] \qquad (6.40)$$
$$f_{02} = 31\,908\,126.78 \pm 0.94 \text{ kHz} \qquad [\,29 \text{ ppb}]. \qquad (6.41)$$

The figure shows good agreement between measurements of the largest intervals; these are best for determining α. The measurements of the small interval also agree well. This interval is less useful for determining α but is a useful check on the theory.

Because a fine structure interval frequency f goes as $R_\infty \alpha^2$ to lowest order, and the Rydberg is known much more accurately than α, a fractional uncertainty in f translates into a fractional uncertainty for α that is smaller by half – if the theory would contribute no additional uncertainty. The 24 ppb fractional uncertainty in the f_{01} that we reported back in 2005 would then suffice to determine α to 12 ppb, a small enough uncertainty to allow this value to be plotted with the most precise measurements in Fig. 6.1.

A big disappointment is that Fig. 6.4 reveals two serious problems with calculations done independently by two different groups [5, 6]. (See Chapter 7.) The calculated interval frequencies (using α from $g/2$) are plotted below the measurements in the figure. The first problem is that the two

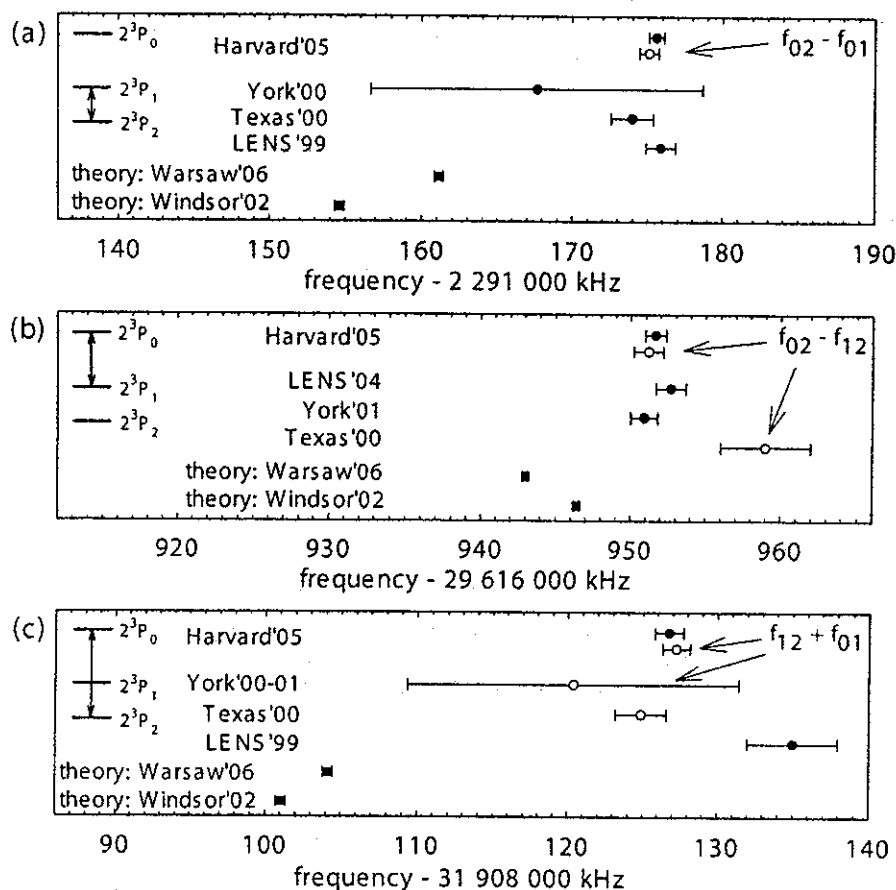212   *G. Gabrielse*

Fig. 6.4. Most accurate measured [1–4] and calculated [5, 6] ^{4}He fine structure intervals with standard deviations. Directly measured intervals (black filled circles) are compared to indirect values (open circles) deduced from measurements of the other two intervals. Uncorrelated errors are assumed for the indirect values for other groups.

calculations do not agree, raising questions as to whether mistakes have been made. It is not hard to imagine mistakes given that the two-electron QED theory gives interval frequencies that are the sum of a series in powers of both α and $\ln \alpha$. The convergence is not rapid, and the many terms to be summed present a significant bookkeeping challenge. The second problem is that both theories disagree with the measurements, for both the large and small intervals. The measurements from 2005 and earlier, though they have an accuracy that would suffice to be one of the most precise determinations of α, cannot be used until the theory issues are resolved.

A serious difficulty with two-electron QED theory seems surprising given how successful one-electron QED theory has been in its predictions. Is

there a fundamental problem or is this a case of mistakes? Until the two calculations agree the latter explanation is hard to discount, and neither calculation agrees with experiments.

A problem with the measurements is the other possibility, though the good agreement between measurements with very different systematic effects would suggest otherwise. One caution is that the most accurate measurements determine to 700 Hz the center of resonance lines that are slightly bigger than 1.6 MHz natural linewidths. "Splitting the line" to a few parts in 10^4 of the linewidth is challenging, requiring as it does that systematic shifts and distortions of the measured resonance lines be either insignificant or well understood. It is hard to believe that a helium fine structure measurement could ever approach the accuracy of the current α from $g/2$.

After we published our measurement of the helium fine structure intervals we narrowed our laser linewidth to below 5 kHz and stabilized it to an iodine clock using an optical comb that we built to bridge between the very different frequencies of our clock and the 1.08 μm optical transitions that we measured. We also greatly improved the signal-to-noise ratio in our measured resonance lines. Within a couple of hours we could get close to 100 Hz resolution for all three intervals, and we could do this in an automated way during the mechanically and electrically quiet nighttimes with none of us present.

However, at the new level of precision that we were exploring we encountered systematic frequency shifts that suggested to us that we had pushed saturated absorption measurements in a discharge cell as far as they should reasonably be pushed. Given the large amount of line splitting already being done, and the theoretical inconsistencies, we decided not to replace the cell with a helium beam. Instead, several years ago we shut the experiment down – perhaps the first discontinued optical comb experiment – and decided to pursue measurements of the electric dipole moment of the electron instead.

6.5.2. *Historically important methods*

In Fig. 6.1 there is a factor of more than 20 between the sizes of the uncertainties for the most accurate determinations of α that have already been discussed above. All other measurements of α have larger error bars that will not fit on this scale. Several additional measurements fit on the 8 times expanded scale of Fig. 6.5, though the error bars for the most accurate determinations of α from $g/2$ are then too small to be visible.
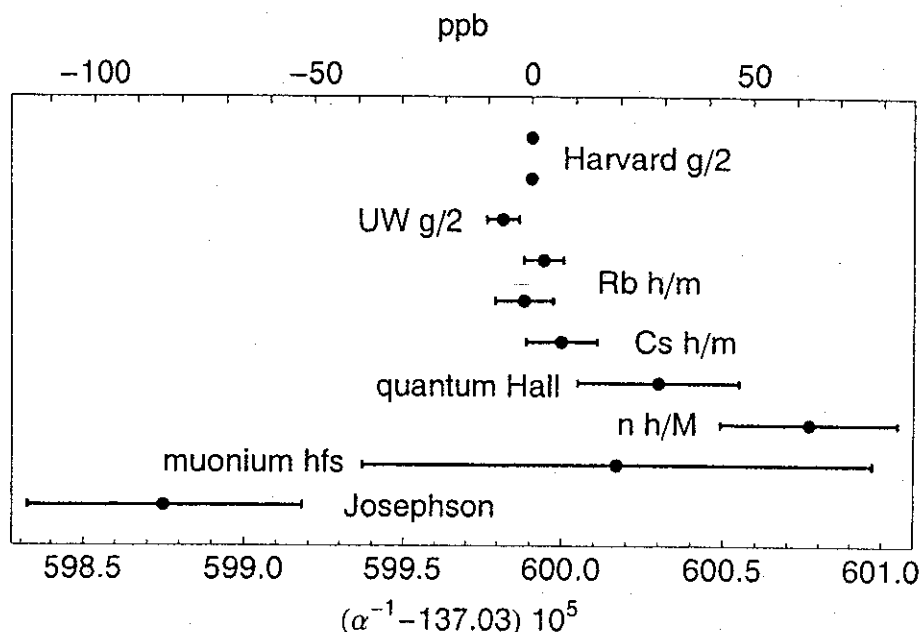
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ppb

Fig. 6.5. Less accurate measurements of α compared upon an expanded scale. The
uncertainties in the two most accurate determinations of α are too small to be visible
on this large scale.

A summary and discussion of traditional measurements of α is in
Ref. [23].   The work includes the value deduced from the quantum Hall
resistance [64], a value that essentially agrees with the more accurate deter-
minations of α insofar as these lie almost within its one standard deviation
error bars.

A measurement using neutrons [65] that is similar in spirit to the de-
scribed Cs and Rb measurements is also plotted. Different mass ratios are
required, of course, but an even more important difference is that h/M_n is
deduced from the diffraction of cold neutrons from a Si crystal. The lattice
spacing in Si is thus crucial, and there is an impressive range of differing
values for this lattice constant [23]. A recommended value [23] is used for
the figure but given the range of measured lattice constants it is not so
surprising that this value of α does not agree so well with more accurate
measurements.

Values from muonium hyperfine structure measurements [23, 43] and
from measurements of the AC Josephson effect (with related measure-
ments [23]) are also plotted because of their importance in the past. It is
not clear why the latter solid state measurement disagree so much with the
more accurate values.

6.6. Conclusion

Combined measurements of the Rydberg constant, two mass ratios, a laser frequency, and an atom recoil frequency together determine α using Cs atoms to 8.0 ppb, and using Rb atoms to 4.6 ppb. Efforts are underway to improve both sets of measurements enough to determine α to 1 ppb.

Helium fine structure measurements are now accurate enough to determine α at nearly the same precision, but with completely different systematic uncertainties. Unfortunately, the two-electron QED theory needed to relate fine structure intervals to α heeds to be clarified before this can happen.

New measurements of the electron magnetic moment $g/2$, along with QED calculations, determine the fine structure constant much more accurately than ever before, to 0.4 ppb. The uncertainty in α will be reduced, without the need for a more accurate measurement of $g/2$, when a first calculation of the tenth-order QED coefficient is completed. It seems reasonable to reduce the experimental and theoretical contribution to determinations of α from $g/2$ to 0.1 ppb or better in efforts now underway, though this will take some time.

Acknowledgments

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